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ON THE CONGRUENCE LATTICE CHARACTERIZATION THEOREM[1]

BY

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ABSTRACT. A simplified proof is given for the theorem characterizing the congruence lattice of a universal algebra.

1. **Introduction.** G. Birkhoff and O. Frink observed that the congruence lattice of a universal algebra was an algebraic lattice [1]. G. Grätzer and E. T. Schmidt proved the converse [3]. Through some rather ingenious methods they built a universal algebra with given congruence lattice; but their proof (and the proof in [2]) that the algebra they constructed had the required congruence lattice was quite difficult. It used a very lengthy and repetitious series of direct computations. E. T. Schmidt and the author independently arrived at essentially the same abstraction of these computations [9], [6, Chapter 2]. (The author announced a crude form of the primary lemma of that abstraction in Abstract #68T-A8 in the Notices Amer. Math. Soc. 15 (1968), 783.)

Recently, however, the author observed a property of the main part of the Grätzer-Schmidt construction that had not been utilized. In this paper we will use this property and give a new proof of this Grätzer-Schmidt representation theorem. The proof given here using this property is simpler than any that has appeared before.

All the previous techniques used for such a proof are valid only for unary algebras. In addition, the only application [9] to a new result of the technique developed in [9] or [6, Chapter 2] was incorrect.

However, the technique presented here is applicable to nonunary algebras. That, and other aspects of this technique, will be used in a series of papers under preparation to settle several problems concerning congruence lattices of universal algebras.

The nonunary applications of this technique are complex. They are easier to follow if one has read the unary case presented here first.

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The primary lemma having to do with the recently observed property is in §4. The property is condition (C) of Lemma 8. In §5 the "three-leaf" construction of Grätzer and Schmidt is thoroughly examined. It is shown there that the construction has the new property.

Next we give a "road map" for the proof.

Let $\mathfrak{L}$ be an algebraic lattice. In order to build the algebra $\mathfrak{A}$ whose congruence lattice is isomorphic to $\mathfrak{L}$, we start with a unary algebra $\mathfrak{B}_0$ and a family $\mathcal{H}_0$ of congruence relations of $\mathfrak{B}_0$. We will arrange things so that $\langle \mathcal{H}_0; \subseteq \rangle$ is isomorphic to $\mathfrak{L}$. Thus we would like to get rid of the congruence relations of $\mathfrak{B}_0$ that do not belong to $\mathcal{H}_0$. So we will add some more structure to $\mathfrak{B}_0$ forming a unary partial algebra $\mathfrak{B}_0^+$. We turn this into an algebra $\mathfrak{B}_1$ by taking $\mathbf{F}(\mathfrak{B}_0^+)$, the algebra "freely generated by $\mathfrak{B}_0^+$". Unfortunately $\mathfrak{B}_1$ has far too many congruence relations. The best we can do is to find in a natural way a family $\mathcal{H}_1$ of congruence relations of $\mathfrak{B}_1$ such that $\langle \mathcal{H}_1; \subseteq \rangle$ is isomorphic to $\mathfrak{L}$. $\mathcal{H}_1$ will be a system of extensions of members of $\mathcal{H}_0$. One repeats the process, thus forming the sequence $\langle \mathfrak{B}_0, \mathcal{H}_0 \rangle, \cdots, \langle \mathfrak{B}_n, \mathcal{H}_n \rangle, \cdots$. We take $\mathfrak{A}$ to be the "direct union" of the $\mathfrak{B}_i$. If everything goes alright, the congruence lattice of $\mathfrak{A}$ will be isomorphic to $\mathfrak{L}$.

Let $\mathfrak{B}$ be a partial algebra, and let Θ and Φ be congruence relations of $\mathfrak{B}$. From the above paragraph the reader can see that it would be important to know that there is some congruence relation on $\mathbf{F}(\mathfrak{B})$, the algebra freely generated by $\mathfrak{B}$, which extends Θ. The reader can imagine that it would be helpful to have an explicit description of $\mathbf{F}(\Theta)$, the smallest congruence relation of $\mathbf{F}(\mathfrak{B})$ extending Θ. §2 is devoted to giving an explicit description of both $\mathbf{F}(\mathfrak{B})$ and $\mathbf{F}(\Theta)$.

All the difficulties in the proof are related to the question of when does it happen that $\mathbf{F}(\Theta \cap \Phi) = \mathbf{F}(\Theta) \cap \mathbf{F}(\Phi)$. (Clearly this is an important question because the intersection of any family of congruence relations is again a congruence relation.) The lemmas of §4 give sufficient conditions for $\mathbf{F}(\Theta \cap \Phi) = \mathbf{F}(\Theta) \cap \mathbf{F}(\Phi)$. §5 gives the construction for passing from $\mathfrak{B}_n$ to $\mathfrak{B}_n^+$. This construction is rather complicated because one has to work hard to "kill" the excess congruences while simultaneously setting things up so that the conditions given in §4 for $\mathbf{F}(\Theta \cap \Phi) = \mathbf{F}(\Theta) \cap \mathbf{F}(\Phi)$ are also satisfied. §3 provides us with the means for taking care of some of the technical details that arise in §4 and §5.

Now we make a few comments about notational usage. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are abelian groups on the sets A and B respectively. It is common practice to use $+$ to denote the operation on each group. Technically this practice is incorrect because on the one hand $+$ is a mapping from $A \times A$ into A and on the other hand it is a mapping from $B \times B$ into B. Some of the technical difficulties are resolved if one thinks of $+$ as the "name" for the operation. The role of $+$ and $\cdot$ as "names" becomes more important in definitions within ring theory. In this paper we will follow this customary algebraic practice of blurring the distinction between an operation and a "name" of an operation. We will, for example, denote an algebra by $\mathfrak{A}$ or by $\langle A;\ F\rangle$. F will be thought of as a set of operations on the set A. But at other times it would be more appropriate to view F as a set of "names" for operations. For example, another algebra whose operations have the same "names" as those in $\mathfrak{A}$ might be denoted by $\langle B;\ F\rangle = \mathfrak{B}$. Also the statement $f \in F$ will at times mean f is an operation on the set A belonging to F. At other times it will be more appropriate to interpret $f \in F$ as f is an "operation name" belonging to the set of names F.

The terminology essentially conforms to that of [2]. $\mathcal{C}(\mathfrak{B})$ will denote the system of congruence relations on the partial algebra $\mathfrak{B}$. $\mathfrak{C}(\mathfrak{B}) = \langle \mathcal{C}(\mathfrak{B});\ \subseteq\rangle$ is the congruence lattice. $D(f,\ \mathfrak{B})$ denotes the domain of the partial operation f in the partial algebra $\mathfrak{B}$.

This new proof of the representation theorem is given without much further comment in §2–§6. §7 contains a further exposition of various aspects of the proof and other comments.

The reader may find it helpful to read §2–§6 lightly the first time. Then read §7, and then reread §2–§6 more thoroughly.

2. **Extensions of congruences.** The lemmas of this section are well known and no proofs will be given.

We start with some definitions. Let $\mathfrak{B} = \langle B;\ F\rangle$ be a partial unary algebra; and let $f \in F$. $\mathfrak{B}[f] = \langle B[f];\ F\rangle$, called $\mathfrak{B}$ (freely) extended by f, is a partial (unary) algebra with the following properties:

(i) $\mathfrak{B}$ is a relative subalgebra of $\mathfrak{B}[f]$ (see p. 80 of [2]);

(ii) $B[f] = B \cup \{f(x) \mid x \in B\}$;

(iii) for any $g \in F$, if $g(y)$ is defined, then $y \in B$;

(iv) if $g \in F$ and $f(x) = g(y) \notin B$, then $g = f$ and $x = y$.

Remark. Clearly one can build $\mathfrak{B}[f]$ by simply adding a new point to B, which is to be $f(x)$, for each $x \in B - D(f,\ \mathfrak{B})$.

$\mathfrak{B}[F] = \langle B[F];\ F\rangle$, called $\mathfrak{B}$ (freely) extended by F, is a partial (unary) algebra with the following properties:

(i) $\mathfrak{B}$ is a relative subalgebra of $\mathfrak{B}[F]$;

(ii) $B[F]$ is generated by B;

(iii) if $f \in F$, then $D(f,\ \mathfrak{B}[F]) = B$;

(iv) if $f_0, f_1 \in F$ and $f_0(x) = f_1(y) \notin B$, then $f_0 = f_1$ and $x = y$.

Remark. One can build $\mathfrak{B}[F]$ is a way similar to that for $\mathfrak{B}[f]$. Here one adds a new point for each $x \in B - D(f,\ \mathfrak{B})$, and one adds these new points for each $f \in F$.

$\mathbf{F}(\mathfrak{B}) = \langle \mathbf{F}(B);\ F\rangle$, called the algebra freely generated by $\mathfrak{B}$, is an algebra with the following properties:

(i) $\mathfrak{B}$ is a relative subalgebra of $\mathbf{F}(\mathfrak{B})$;

(ii) $\mathbf{F}(\mathfrak{B})$ is generated by B;

(iii) if $\mathfrak{A}$ is any algebra and ϕ is any homomorphism from $\mathfrak{B}$ into $\mathfrak{A}$, then ϕ has an extension to a homomorphism from $\mathbf{F}(\mathfrak{B})$ into $\mathfrak{A}$.

For the lemmas of this section let $\mathfrak{B} = \langle B; F\rangle$ be any partial unary algebra, and let $f \in F$. Set $\mathfrak{B} = \mathfrak{B}[F]^0$, and set $\mathfrak{B}[F]^{n+1} = (\mathfrak{B}[F]^n)[F] = \langle B[F]^{n+1}; F\rangle$.

Lemma 1. *$\mathfrak{B}[f]$ and $\mathfrak{B}[F]$ and $\mathbf{F}(\mathfrak{B})$ all exist. Moreover, $B[F] = \bigcup(B[f] \mid f \in F) = B \dot\cup \dot\bigcup(B[f] - B \mid f \in F)$. In addition $\mathbf{F}(B) = \bigcup(B[F]^n \mid n = 0, 1, \cdots)$.*

Let Θ be an equivalence relation on the set B and let Φ be an equivalence relation on C and let $B \subseteq C$. Φ is an *extension* of Θ iff $\Theta = \Phi\mid_B = \Phi \cap B^2$.

The next three lemmas are concerned with extending congruences of $\mathfrak{B}$ to $\mathfrak{B}[f]$, $\mathfrak{B}[F]$ and $\mathbf{F}(\mathfrak{B})$.

Lemma 2. *Let Θ be a congruence of $\mathfrak{B}$. There is a smallest extension $\Theta[f]$ of Θ to a congruence of $\mathfrak{B}[f]$. Moreover, for $x, y \in B[f]$, $x \equiv y$ $(\Theta[f])$ iff one of the following holds:*

(i) *$x, y \in B$ and $x \equiv y$ (Θ);*

(ii) *$x \in B$, $y = f(s) \notin B$, and there exists $u = f(t) \in B$ with $x \equiv u$ (Θ) and $s \equiv t$ (Θ);*

(iii) *$x = f(s) \notin B$, $y \in B$ and the condition symmetric to (ii) holds;*

(iv) *$x = f(s) \notin B$ and $y = f(t) \notin B$ and one of the following holds: (iv_1) $s \equiv t$ (Θ); (iv_2) there exist $u = f(r) \in B$ and $v = f(w) \in B$ such that $r \equiv s$ (Θ), $u \equiv v$ (Θ) and $w \equiv t$ (Θ).*

Lemma 3. *Let Θ be a congruence of $\mathfrak{B}$. There is a smallest extension $\Theta[F]$ of Θ to a congruence of $\mathfrak{B}[F]$. Moreover, for $x, y \in B[F]$, $x \equiv y$ $(\Theta[F])$ iff one of the following holds:*

(i) *$x, y \in B[f]$ for some f and $x \equiv y$ $(\Theta[f])$;*

(ii) *$x \in B[f_0]$, $y \in B[f_1]$ and there exists a $z \in B$ with $x \equiv z$ $(\Theta[f_0])$ and $z \equiv y$ $(\Theta[f_1])$.*

Lemma 4. *Let Θ be a congruence of $\mathfrak{B}$. There is a smallest extension $\mathbf{F}(\Theta)$ of Θ to a congruence of $F(\mathfrak{B})$. Moreover, if $\Theta = \Theta_0$ and $\Theta_{n+1} = \Theta_n[F]$, then $\mathbf{F}(\Theta) = \bigcup(\Theta_n \mid n = 0, 1, \cdots)$.*

3. **Extensions of systems of congruences.** Again we start with a definition. Suppose $\mathcal{H}$ is a system (set) of equivalence relations on the set B. $\mathcal{H}$ is a *unary-algebraic* closure system iff there exists an F such that $\mathcal{H}$ is the system of all congruences of the unary partial algebra $\langle B; F\rangle$.

ω denotes the equality relation. If $\mathcal{H}$ is a closure system of equivalence relations on the set B and $a, b \in B$, then $\Theta_{\mathcal{H}}(a, b)$ will denote the smallest member of $\mathcal{H}$ under which a is equivalent to b; i.e., $\Theta_{\mathcal{H}}(a, b)$ is the closure in $\mathcal{H}$ of $\langle a, b\rangle$.

Lemma 5. *A closure system $\mathcal{H}$ of equivalence relations on the set B is unary-algebraic iff $\omega \in \mathcal{H}$ and for every equivalence relation Θ on B $\Theta_{\mathcal{H}}(a, b) \subseteq \Theta$ for all $\langle a, b\rangle \in \Theta$ implies $\Theta \in \mathcal{H}$.*

Proof. That every unary-algebraic closure system satisfies the stated condition is obvious. So let $\mathcal{H}$ be a closure system of equivalence relations on the set B. Assume that $\omega \in \mathcal{H}$ and that, for every equivalence relation Θ on B, $\Theta_{\mathcal{H}}(a, b) \subseteq \Theta$ for all $\langle a, b\rangle \in \Theta$ implies $\Theta \in \mathcal{H}$. Let $\Lambda = \{\langle a, b, c, d\rangle \mid c \equiv d\ (\Theta_{\mathcal{H}}(a, b))\}$. For each $\lambda = \langle a, b, c, d\rangle \in \Lambda$ define a partial unary operation f_λ with $D(f_\lambda) = \{a, b\}$ and $f_\lambda(a) = c$ and $f_\lambda(b) = d$. That $\omega \in \mathcal{H}$ implies f_λ is well defined. Clearly the assumptions on $\mathcal{H}$ imply that $\mathcal{H}$ is the system of all congruences of $\langle B; \{f_\lambda \mid \lambda \in \Lambda\}\rangle$.

We need another definition. Let $\mathfrak{B} = \langle B; F\rangle$ and $\mathfrak{C} = \langle C; G\rangle$ be partial unary algebras. $\mathfrak{C}$ is an *expansion* of $\mathfrak{B}$ iff $B \subseteq C$ and $F \subseteq G$ and if $f(b)$ is defined in $\mathfrak{B}$ then $f(b)$ is defined and has the same value in $\mathfrak{C}$. (Technically, we do not mean F is a subset of G, but we mean every name of an operation in $\mathfrak{B}$ is the name of an operation in $\mathfrak{C}$.)

For the next two lemmas suppose that the following are true. The unary partial algebra $\mathfrak{C}$ is an expansion of the unary partial algebra $\mathfrak{B}$. $\mathcal{H}$ is a system of congruence relations of $\mathfrak{B}$. If $\Theta \in \mathcal{H}$, then there exists a smallest extension Θ^* of Θ to a congruence of $\mathfrak{C}$. $\mathcal{H}^* = \{\Theta^* \mid \Theta \in \mathcal{H}\}$.

Lemma 6. *If $\mathcal{H}$ is a unary-algebraic closure system and $\mathcal{H}^*$ is a closure system, then $\mathcal{H}^*$ is a unary-algebraic closure system.*

Proof. Let Θ be an equivalence relation on C, and suppose that for every $\langle a, b\rangle \in \Theta$, $\Theta_{\mathcal{H}^*}(a, b) \subseteq \Theta$. Since $\Theta_{\mathcal{C}(\mathfrak{C})}(a, b) \subseteq \Theta_{\mathcal{H}^*}(a, b)$ for every $a, b \in C$ and since by definition $\mathcal{C}(\mathfrak{C})$ is unary-algebraic, it follows that Θ is a congruence of $\mathfrak{C}$. Let $\Phi = \Theta \cap B^2$ and let $\langle a, b\rangle \in \Phi$. Since $\Theta_{\mathcal{H}^*}(a, b) = (\Theta_{\mathcal{H}}(a, b))^* \subseteq \Theta$, it follows that $\Theta_{\mathcal{H}}(a, b) \subseteq \Phi$. Since $\mathcal{H}$ is unary-algebraic, by Lemma 5 $\Phi \in \mathcal{H}$. Since Θ is a congruence, $\Phi^* \subseteq \Theta$. Let $\langle x, y\rangle \in \Theta$. There is a $\Psi \in \mathcal{H}$ such that $\Psi^* = \Theta_{\mathcal{H}^*}(x, y)$. Clearly $\Psi \subseteq \Theta$. Since $\Xi \to \Xi^*$ is an order isomorphism, $\langle x, y\rangle \in \Psi^* \subseteq \Phi^*$, and so $\Theta \subseteq \Phi^*$. Thus $\Theta = \Phi^* \in \mathcal{H}^*$. Since $\omega \in \mathcal{H}$ and $\omega^* = \omega$, $\mathcal{H}$ is unary-algebraic by Lemma 5.

Lemma 7. *If $\mathcal{H}$ is a closure system and if for every $\{a, b\} \subseteq C$ there exists a member of $\mathcal{H}$, denoted by $\Phi(a, b)$, satisfying the following:*

(A) $a \equiv b\ ((\Phi(a, b))^*)$,

(B) $a \equiv b\ (\Theta^*)$ *and* $\Theta \in \mathcal{H}$ *imply* $\Phi(a, b) \subseteq \Theta$,

then $\mathcal{H}^*$ is a closure system.

Proof. Let $(\Theta_i^* \mid i \in I)$ be a family of members of $\mathcal{H}^*$, and suppose $a \equiv b\ (\bigcap(\Theta_i^* \mid i \in I))$. So $a \equiv b\ (\Theta_i^*)$ for all i, and $\Phi(a, b) \subseteq \Theta_i$ for all i. So $\Phi(a, b) \subseteq \bigcap(\Theta_i \mid i \in I)$, and we get that $a \equiv b\ ((\bigcap(\Theta_i \mid i \in I))^*)$. Now suppose $a \equiv b\ ((\bigcap \Theta_i \mid i \in I)^*)$. So $\Phi(a, b) \subseteq \bigcap(\Theta_i \mid i \in I)$, and $\Phi(a, b) \subseteq \Theta_i$ for each i. Thus $a \equiv b\ (\Theta_i^*)$ for each i, and we now have $a \equiv b\ (\bigcap(\Theta_i^* \mid i \in I))$. So $\bigcap(\Theta_i^* \mid i \in I) = (\bigcap(\Theta_i \mid i \in I))^* \in \mathcal{H}^*$, and $\mathcal{H}^*$ is a closure system.

4. $\mathbf{F}(\mathcal{H})$ can be a closure system. There are three lemmas in this section having to do with the "new" property.

Let $\mathcal{H}[F] = \{\Theta[F] \mid \Theta \in \mathcal{H}\}$ and $\mathbf{F}(\mathcal{H}) = \{\mathbf{F}(\Theta) \mid \Theta \in \mathcal{H}\}$.

Lemma 8. *Let $\mathfrak{B} = \langle B; F\rangle$ be a unary partial algebra and let $\mathcal{H}$ be a closure system of congruences of $\mathfrak{B}$. If*

(A) *there is a $C \subseteq B$, $C \neq \emptyset$, such that $D(f, \mathfrak{B}) = C$ for all $f \in F$,*

(B) *there is a function $\gamma: B \to C$ so that $\Theta_{\mathcal{H}}(b, \gamma(b)) \subseteq \Theta_{\mathcal{H}}(b, c)$ for all $c \in C$,*

(C) *for any $a, b \in B$ and $f \in F$ and $\Theta \in \mathcal{H}$ it holds that $f(a) \equiv f(b)\ (\Theta)$ implies $a \equiv b\ (\Theta)$,*

then $\mathcal{H}[F]$ is a closure system and $\mathcal{H}[F]$ and $\mathfrak{B}[F]$ satisfy (A)–(C).

First we need another lemma.

Lemma 9. *Under the hypotheses of Lemma 8, for every $\{a, b\} \subseteq B[F]$ there is a $\Phi(a, b) \in \mathcal{H}$ satisfying conditions* (A) *and* (B) *of Lemma 7. Moreover*

(i) *if $\{a, b\} \subseteq B$, then $\Phi(a, b) = \Theta_{\mathcal{H}}(a, b)$;*

(ii) *if $a \in B$ and $b = f(d) \in B[F] - B$, then $\Phi(a, b) = \Theta_{\mathcal{H}}(a, f(\gamma(d))) \vee \Theta_{\mathcal{H}}(\gamma(d), d)$;*

(iii) *if $a = f(c)$, $b = f(d) \in B[F] - B$, then $\Phi(a, b) = \Theta_{\mathcal{H}}(c, d)$;*

(iv) *if $a = f(c)$, $b = g(d) \in B[F] - B$ and $f \neq g$, then $\Phi(a, b) = \Theta_{\mathcal{H}}(c, \gamma(c)) \vee \Theta_{\mathcal{H}}(f(\gamma(c)), g(\gamma(d))) \vee \Theta_{\mathcal{H}}(\gamma(d), d)$.*

Remark. The reader should derive from context within which lattice the $\vee$ is being taken. For example in the formulas above $\vee$ is in $\langle \mathcal{H}; \subseteq\rangle$.

Proof. Let $\{a, b\} \subseteq B[F]$ and let $\Psi(a, b)$ be the member of $\mathcal{H}$ given by the appropriate formula from (i)–(iv). Since $C \neq \emptyset$, $\Psi(a, b)$ exists. Clearly $a \equiv b\ ((\Psi(a, b))[F])$. So let $\Theta \in \mathcal{H}$ with $a \equiv b\ (\Theta[F])$.

If $\{a, b\} \subseteq B$, then $a \equiv b\ (\Theta)$, and so $\Theta_{\mathcal{H}}(a, b) \subseteq \Theta$.

Suppose $a \in B$ and $b = f(d) \in B[F] - B$. Then by (3.i) and (2.ii) there exists $u = f(t) \in B$ with $a \equiv u\ (\Theta)$ and $d \equiv t\ (\Theta)$. By hypothesis $t \equiv d \equiv \gamma(d)\ (\Theta)$. So $a \equiv u = f(t) \equiv f(\gamma(d))\ (\Theta)$. (ii) now follows.

Suppose $a = f(c)$ and $b = f(d)$ are in $B[F] - B$. Suppose there exist $u = f(r) \in B$ and $v = f(w) \in B$ with $c \equiv r\ (\Theta)$ and $u \equiv v\ (\Theta)$ and $w \equiv d\ (\Theta)$. By hypothesis $r \equiv w\ (\Theta)$, and so $c \equiv d\ (\Theta)$. (iii) now follows from (3.i) and (2.iv).

Suppose $a = f(c)$ and $b = g(d)$ are in $B[F] - B$, and suppose $f \neq g$. By (3.ii) there is a $z \in B$ with $a \equiv z$ $(\Theta[F])$ and $z \equiv b$ $(\Theta[F])$. So by (ii) the following are congruent under Θ, $c \equiv \gamma(c)$ and $f(\gamma(c)) \equiv z$ and $z \equiv g(\gamma(d))$ and $\gamma(d) \equiv d$. So $f(\gamma(c)) \equiv g(\gamma(d))$ (Θ), and (iv) follows.

Proof of Lemma 8. That $\mathfrak{H}[F]$ is a closure system follows from Lemmas 7 and 9. That (A) holds is obvious because $D(f, \mathfrak{B}[F]) = B$ for all $f \in F$ by definition. Define γ': $B[F] \to B$ by $\gamma'(b) = b$ if $b \in B$ and $\gamma'(f(b)) = f(\gamma(b))$ for $f(b) \in B[F] - B$. Clearly γ' is well defined, and from (9.ii) it follows that (B) holds. Let $a, b \in B$, let $f \in F$, let $\Theta \in \mathfrak{H}$ and suppose $f(a) \equiv f(b)$ $(\Theta[F])$. If $\{f(a), f(b)\} \subset B$ or $\{f(a), f(b)\} \subseteq B[F] - B$, then by (C) and (9.i) and (9.iii) $a \equiv b$ $(\Theta[F])$. Suppose $f(a) \in B$ and $f(b) \in B[F] - B$. Then by (9.ii) $f(a) \equiv f(\gamma(b))$ (Θ) and $\gamma(b) \equiv b$ (Θ). By (C) $a \equiv \gamma(b) \equiv b$ (Θ).

The purpose of this section has been to obtain the following lemma.

Lemma 10. *Under the hypotheses of Lemma* 8, $\mathbf{F}(\mathfrak{H})$ *is a closure system. Moreover, if* $a, b \in \mathbf{F}(B)$ *and* $f \in F$ *and* $\Theta \in \mathbf{F}(\mathfrak{H})$, *then* $f(a) \equiv f(b)$ (Θ) *implies* $a \equiv b$ (Θ).

Proof. This follows from Lemmas 8 and 4. For example, let $\Phi \in \mathfrak{H}$, let $\Theta = \mathbf{F}(\Phi)$, let $a, b \in \mathbf{F}(B)$ and let $f(a) \equiv f(b)$ (Θ). So by Lemma 4 there is an n such that $f(a) \equiv f(b)$ (Φ_n). It follows from Lemma 8 that $a \equiv b$ (Φ_n).

5. **Concerning the three-leaf construction.** Let Φ be a congruence of the partial algebra $\mathfrak{A}$. For each congruence Θ of $\mathfrak{A}$ with $\Theta \supseteq \Phi$ define a relation on A/Φ as follows:

$$[a]\Phi \equiv [b]\Phi \ (\Theta/\Phi) \ \text{ iff } a \equiv b \ (\Theta).$$

As is well known, the congruences of $\mathfrak{A}/\Phi$ are exactly the relations of the form Θ/Φ, and $\Theta \to \Theta/\Phi$ is an isomorphism from the dual ideal generated by Φ in the congruence lattice of $\mathfrak{A}$ onto the congruence lattice of $\mathfrak{A}/\Phi$.

In addition to its use below as a superscript, + will also be used to denote the join in the lattice of equivalence relations.

Lemma 11. *Suppose* $\mathfrak{H}$ *is a unary-algebraic closure system of congruences of the unary algebra* $\mathfrak{B} = \langle B; F\rangle$, *and suppose* $f(a) \equiv f(b)$ (Θ) *implies* $a \equiv b$ (Θ) *for any* $a, b \in B$, $f \in F$ *and* $\Theta \in \mathfrak{H}$. *There exists a partial unary algebra* $\mathfrak{B}^+ = \langle B^+; F^+\rangle$ *such that*

(i) $\mathfrak{B}^+$ *is an expansion of* $\mathfrak{B}$;

(ii) *the only congruences of* $\mathfrak{B}$ *that have extensions to congruences of* $\mathfrak{B}^+$ *are members of* $\mathfrak{H}$;

(iii) *each member* Θ *of* $\mathfrak{H}$ *has a smallest extension* Θ^+ *to a congruence of* $\mathfrak{B}^+$;

(iv) $\mathfrak{H}^+ = \{\Theta^+ \mid \Theta \in \mathfrak{H}\}$ *is a closure system*;

(v) $\mathfrak{B}^+$ *and* $\mathfrak{H}^+$ *satisfy the hypotheses of Lemma* 8.

Remark. The proof of this lemma uses the Grätzer-Schmidt "three-leaf" construction.

Proof. Let $\Lambda = \{\langle a, b, c, d\rangle | \, c \equiv d \;(\Theta_{\mathcal{H}}(a, b)), a \neq b, c \neq d\}$. For each $\lambda \in \Lambda$ and for each $i \in \{1, 2, 3\}$ let $f_{\lambda,i}$ be a unary partial operation with $D(f_{\lambda,i}) = \varnothing$. Set $F^+ = F \cup \{f_{\lambda,i} | \, \lambda \in \Lambda, i \in \{1, 2, 3\}\}$, and set $\mathfrak{B}' = \langle B; F^+\rangle$. Consider $\mathfrak{B}'[F^+] = \langle B[F^+], F^+\rangle$, and notice that $B[F^+] = B[\{f_{\lambda,i} | \, \lambda \in \Lambda, i \in \{1, 2, 3\}\}]$. For all $\lambda \in \Lambda$ let $f_{\lambda,0}$ be the identity function on B. Set $a_0 = d$, $b_0 = c$, and $a_i = a$, $b_i = b$ for $i = 1, 2, 3$. Define the relation Ψ on $B[F^+]$ by $x \equiv y \;(\Psi)$ iff $x = y$ or $\{x, y\} = \{f_{\lambda,i}(b_i), f_{\lambda,i+1}(a_{i+1})\}$ for some $\lambda \in \Lambda$ and some $i \in \{0, 1, 2, 3\}$. The addition in the subscripts means addition modulo 4. Note for fixed λ that Ψ restricted to $B \cup \bigcup (f_{\lambda,i}(B) | \, i = 1, 2, 3)$ is an equivalence relation. Let Φ be the transitive closure of Ψ. Φ is an equivalence relation. Observe that Φ restricted to $B \cup \bigcup (f_{\lambda,i}(B) | \, i = 1, 2, 3)$ equals Ψ restricted to the same set. Alternatively one can define Φ to be the equivalence relation on $B[F^+]$ defined by $x \equiv y \;(\Phi)$ iff one of the following holds:

(a) $x = y$;

(b) $\{x, y\} = \{f_{\lambda,i}(b_i), f_{\lambda,i+1}(a_{i+1})\}$ for some $\lambda \in \Lambda$ and some $i \in \{0, 1, 2, 3\}$;

(c) there exists $\lambda = \langle a, b, c, d\rangle$ and $\mu = \langle e, f, g, h\rangle$ such that one of the following holds:

(c_1) $c = g$ and $\{x, y\} = \{f_{\lambda,1}(a), f_{\mu,1}(e)\}$;

(c_2) $c = h$ and $\{x, y\} = \{f_{\lambda,1}(a), f_{\mu,3}(f)\}$;

(c_3) $d = h$ and $\{x, y\} = \{f_{\lambda,3}(b), f_{\mu,3}(f)\}$.

Since the intersection of each equivalence class with the domain of any operation (domain = B) is at most one element, Φ is a congruence relation of $\mathfrak{B}'[F^+]$.

Set $\mathfrak{B}^+ = \langle B^+; F^+\rangle = \mathfrak{B}'[F^+]/\Phi = \langle B[F^+]/\Phi; F^+\rangle$.

Since $\Phi |_B = \omega$, for x in B we can identify x with the class containing x in order to get that $\mathfrak{B}^+$ is an expansion of $\mathfrak{B}$.

Let Z be the subset of $B[F^+]$ with $Z = \bigcup ((f_{\lambda,i}(B))^2 | \, \lambda \in \Lambda, i = 0, 1, 2, 3)$. Observe also that $\mathcal{C}(\mathfrak{B}) = \mathcal{C}(\mathfrak{B}')$.

Remark. Our next task is to show that if Θ is a congruence relation of $\mathfrak{B}$, then Θ has an extension to a congruence relation of $\mathfrak{B}^+$ iff $\Theta \in \mathcal{H}$. We will also give an explicit description of Θ^+, the smallest extension of Θ to $\mathfrak{B}^+$. Claims 1–5 are given over to these two tasks. Claim 1 is more general than needed at this point, but the generality proves useful later.

Claim 1. If $\Theta \in \mathcal{H}$ and $\langle x, y\rangle \in Z$, then $x \equiv y \;(\Theta[F^+] + \Phi)$ iff $x \equiv y \;(\Theta[F^+])$.

Certainly, if $x \equiv y \;(\Theta[F^+])$, then $x \equiv y \;(\Theta[F^+] + \Phi)$. So let $x \equiv y \;(\Theta[F^+] + \Phi)$, and let $\langle x, y\rangle \in (f_{\lambda,j}(B))^2$. So there exists a sequence $x = s_0, \cdots, s_n = y$ such that $s_i \equiv s_{i+1} \;(\Theta[F^+])$ or $s_i \equiv s_{i+1} \;(\Phi)$ for $i = 0, \cdots, n-1$. Assume the sequence is of shortest length. Then $s_i \neq s_j$ if $i \neq j$, $s_i \equiv s_{i+1} \;(\Theta[F^+])$ implies $s_{i+1} \equiv$

s_{i+2} (Φ) (if s_{i+2} exists) and $s_i \not\equiv s_{i+1}$ (Φ), and similarly if $s_i \equiv s_{i+1}$ (Φ). Let $\lambda = \langle a, b, c, d\rangle$, and recall that $a_0 = d$, $b_0 = c$ and $a_i = a$ and $b_i = b$ for $i = 1, 2, 3$.

First we assume that $x = s_0 = f_{\lambda,j}(b_j)$, $s_1 = f_{\lambda,j+1}(a_{j+1})$, $y = s_n = f_{\lambda,j}(a_j)$ and that $j = 0, 1$, or 2. Since $s_1 \notin f_{\lambda,j}(B)$, $1 \neq n$. So there exists an s_2 with $s_1 \equiv s_2$ $(\Theta[F^+])$. Since $D(f_{\lambda,i}, \mathfrak{B}') = \emptyset$ for $i = 1, 2, 3$ and for any λ, it follows from Lemmas 3 and 2 that $s_2 \in f_{\lambda,j+1}(B)$. So $s_2 \notin f_{\lambda,j}(B)$ and $2 \neq n$. So there exists an s_3 with $s_2 \equiv s_3$ (Φ). Since $s_1 \not\equiv s_2 \not\equiv s_3$ and since $j + 1 \neq 0$, it holds that $s_2 = f_{\lambda,j+1}(b_{j+1})$. Since $j + 1 \neq 0$ and $D(f_{\lambda,j+1}, \mathfrak{B}') = \emptyset$, it follows from $f_{\lambda,j+1}(a_{j+1}) = s_1 \equiv s_2 = f_{\lambda,j+1}(b_{j+1})$ $(\Theta[F^+])$ and from Lemmas 3 and 2 that $a = a_{j+1} \equiv b_{j+1} = b$ (Θ). Since $\Theta \in \mathcal{H}$, by definition of Λ, $c \equiv d$ (Θ). So $f_{\lambda,i}(a_i) \equiv f_{\lambda,i}(b_i)$ $(\Theta[F^+])$ for $i = 0, 1, 2, 3$. In particular, $x \equiv y$ $(\Theta[F^+])$.

For the second case we assume that $x = s_0 = f_{\lambda,j}(a_j)$, $s_1 = f_{\lambda,j+3}(b_{j+3})$, $y = s_n = f_{\lambda,j}(b_j)$ and that $j = 0, 2$, or 3. In an argument similar to the first case we prove that $f_{\lambda,j+3}(b_{j+3}) \equiv f_{\lambda,j+3}(a_{j+3})$ $(\Theta[F^+])$. Continuing as in the first case, we find that again $x \equiv y$ $(\Theta[F^+])$.

Now we are ready for the general case. Assume that $n \geq 2$. So there exists an i with $s_i \equiv s_{i+1}$ (Φ). Let l (resp. k) be the smallest (resp. largest) integer with $s_l \equiv s_{l+1}$ (Φ) (resp. $s_k \equiv s_{k-1}$ (Φ)). Recall that $\langle x, y\rangle \in (f_{\lambda,j}(B))^2$. Note that $s_l \neq s_k$. If $j = 1, 2$, or 3, then $s_l \neq s_k$ implies $\{s_l, s_k\} = \{f_{\lambda,j}(a_j), f_{\lambda,j}(b_j)\}$, and it follows easily from the first two special cases that $s_l \equiv s_k$ $(\Theta[F^+])$. So in this case $x \equiv s_l \equiv s_k \equiv y$ $(\Theta[F^+])$, and $x = t_0$, $t_1 = y$ is a sequence shorter than the shortest sequence. So let $j = 0$. Thus for some $\mu = \langle a', b', c', d'\rangle \in \Lambda$, $s_l = f_{\mu,0}(b'_0)$ and $s_{l+1} = f_{\mu,1}(a'_1)$, or $s_l = f_{\mu,0}(a'_0)$ and $s_{l+1} = f_{\mu,3}(b'_3)$. It is easy to show that if $s_l = f_{\mu,0}(b'_0)$, then s_{l+7} exists and $s_{l+7} = f_{\mu,0}(a'_0)$ and that if $s_l = f_{\mu,0}(a'_0)$, then $s_{l+7} = f_{\mu,0}(b'_0)$. So by the first two special cases $s_l \equiv s_{l+7}$ $(\Theta[F^+])$. So in the case $j = 0$, $x = s_0, s_l, s_{l+7}, \cdots, s_n$ is a sequence with appropriate properties and shorter than the shortest such sequence. Since the assumption $n \geq 2$ leads in both cases to our being able to shorten the shortest sequence, $n = 0$ or 1. If $n = 0$, then $x = y$ and so $x \equiv y$ $(\Theta[F^+])$. If $n = 1$, then $x \equiv y$ (Φ) or $x \equiv y$ $(\Theta[F^+])$. If $x \equiv y$ (Φ), then $x, y \in f_{\lambda,j}(B)$ and the definition of Φ imply that $x = y$. So in case any $x \equiv y$ $(\Theta[F^+])$.

Claim 2. For $\Theta \in \mathcal{H}$, $\Theta[F^+] + \Phi$ is an extension of Θ to a congruence $\mathfrak{B}'[F^+]$.

From Claim 1 $\Theta[F^+] + \Phi$ is an extension of Θ to an equivalence relation on $B'[F^+]$. Let $x \equiv y$ $(\Theta[F^+] + \Phi)$, let $f \in F^+$, and suppose $f(x)$ and $f(y)$ are defined. Then $x, y \in B$ and $x \equiv y$ (Θ). So $f(x) \equiv f(y)$ $(\Theta[F^+])$, and in turn $f(x) \equiv f(y)$ $(\Theta[F^+] + \Phi)$.

Claim 3. If a congruence Θ of $\mathfrak{B}$ has an extension to a congruence of $\mathfrak{B}^+$, then $\Theta[F^+] + \Phi$ is an extension of Θ to a congruence of $\mathfrak{B}'[F^+]$, and $\Theta^+ =$

$(\Theta[F^+] + \Phi)/\Phi$ is the smallest extension of Θ to a congruence of $\mathfrak{B}^+$.

Let Θ' be a congruence of $\mathfrak{B}^+$ with $\Theta'|_B = \Theta$. So there exists a congruence Ψ of $\mathfrak{B}'[F^+]$ with $\Phi \subseteq \Psi$ such that $\Psi/\Phi = \Theta'$. So Ψ is an extension of Θ to $\mathfrak{B}'[F^+]$. Since $\Theta[F^+]$ is the smallest extension, $\Theta[F^+] \subseteq \Psi$. Since $\Theta[F^+] + \Phi \subseteq \Psi$, $\Theta[F^+] + \Phi$ is an equivalence relation extending Θ. As in the proof of Claim 2, this implies $\Theta[F^+] + \Phi$ is a congruence of $\mathfrak{B}'[F^+]$. Clearly $\Theta[F^+] + \Phi$ is the smallest extension of Θ to a congruence of $\mathfrak{B}'[F^+]$ bigger than Φ.

Claim 4. If $\Theta \in \mathcal{C}(\mathfrak{B})$ has an extension to a congruence of $\mathfrak{B}^+$, then $a \equiv b$ (Θ) implies $c \equiv d$ (Θ) for all $\langle a, b, c, d\rangle \in \Lambda$.

Let $\lambda = \langle a, b, c, d\rangle \in \Lambda$, let $\Theta \in \mathcal{C}(\mathfrak{B})$ and suppose $a \equiv b$ (Θ). Then $c = b_0 = f_{\lambda,0}(b_0) \equiv f_{\lambda,1}(a_1) = f_{\lambda,1}(a) \equiv f_{\lambda,1}(b) = f_{\lambda,1}(b_1) \equiv f_{\lambda,2}(a_2) \equiv f_{\lambda,2}(b_2) \equiv f_{\lambda,3}(a_3) \equiv f_{\lambda,3}(b_3) \equiv f_{\lambda,0}(a_0) = a_0 = d$ $(\Theta[F^+] + \Phi)$. By Claim 3 $c \equiv d$ (Θ).

Claim 5. $\Theta \in \mathcal{C}(\mathfrak{B})$ has an extension to a congruence of $\mathfrak{B}^+$ iff $\Theta \in \mathcal{H}$.

It follows from Claim 2 that if $\Theta \in \mathcal{H}$, then Θ has an extension to a congruence of $\mathfrak{B}^+$. So let $\Theta \in \mathcal{C}(\mathfrak{B})$ have an extension to a congruence of $\mathfrak{B}^+$. From Claim 4 we get that $\Theta_{\mathcal{H}}(a, b) \subseteq \Theta$ for all $\langle a, b\rangle \in \Theta$ (if $a \neq b$). So by Lemma 5 $\Theta \in \mathcal{H}$.

Remark. The next thing we need to know is that $\mathcal{H}^+ = \{\Theta^+ | \Theta \in \mathcal{H}\}$ is a closure system. In line with Lemma 7 we need to show the existence of the "$\Phi(a, b)$'s". Having explicit descriptions of these $\Phi(a, b)$'s will help us with the remaining details. To that end we have the following claim.

Claim 6. If $x, y \in B^+$, then there exists a $\Phi(x, y) \in \mathcal{H}$ satisfying the conditions of Lemma 7. Moreover,

(a) if $x, y \in B$, then $\Phi(x, y) = \Theta_{\mathcal{H}}(x, y)$;

(b) if $x = f_{\lambda,i}(s)$, $y = f_{\lambda,i}(t)$ and $i = 1, 2$, or 3, then $\Phi(x, y) = \Theta_{\mathcal{H}}(s, t)$;

(c) if $x = f_{\lambda,i}(s)$, $y = f_{\lambda,i+1}(t)$ (i.e., $i + 1 \pmod 4$), then $\Phi(x, y) = \Phi(x, f_{\lambda,i}(b_i)) \vee \Phi(f_{\lambda,i+1}(a_{i+1}), y)$;

(d) if $x \in B$, $y = f_{\lambda,2}(t)$, then $\Phi(x, y) = \Phi(x, f_{\lambda,1}(b_1)) \vee \Phi(f_{\lambda,2}(a_2), y)$;

(e) if $x = f_{\lambda,1}(s)$ and $y = f_{\lambda,3}(t)$, then $\Phi(x, y) = \Phi(x, f_{\lambda,1}(a_1)) \vee \Phi(f_{\lambda,0}(b_0), f_{\lambda,0}(a_0)) \vee \Phi(f_{\lambda,3}(b_3), y)$;

(f) there is a function $\gamma: B^+ \to B$ so that $\Phi(b^+, \gamma(b^+)) \subseteq \Phi(b^+, b)$ for all $b^+ \in B^+$, $b \in B$;

(g) if $x = f_{\lambda,i}(s)$, $y = f_{\mu,j}(t)$ and $\lambda \neq \mu$ and $i \neq 0 \neq j$, then $\Phi(x, y) = \Phi(x, \gamma(x)) \vee \Phi(\gamma(x), \gamma(y)) \vee \Phi(\gamma(y), y)$.

Let $\Psi(x, y)$ denote the member of $\mathcal{H}$ given by the appropriate formula from (a)–(e), (g). Throughout Θ will be a member of $\mathcal{H}$ with $x \equiv y$ (Θ^+).

Let $x, y \in B$. Certainly $x \equiv y$ $((\Psi(x, y))^+)$. On the other hand since Θ^+ is an extension of Θ, $x \equiv y$ (Θ), and $\Psi(x, y) = \Theta_{\mathcal{H}}(x, y) \subseteq \Theta$.

Let $x = f_{\lambda,i}(s)$, $y = f_{\lambda,i}(t)$ and let $i = 1, 2$, or 3. Certainly $x \equiv y$ $((\Psi(x, y))^+)$.

On the other hand, by Claim 3 $x \equiv y$ (Θ^+) implies that, in $B'[F^+]$, $f_{\lambda,i}(s) \equiv f_{\lambda,i}(t)$ $(\Theta[F^+] + \Phi)$. By Claim 1 this implies $f_{\lambda,i}(s) \equiv f_{\lambda,i}(t)$ $(\Theta[F^+])$. Since $D(f_{\lambda,i}, \mathfrak{B}') = \varnothing$, by (3.ii) and (2.iv) this implies that $s \equiv t$ (Θ). So $\Psi(x, y) \subseteq \Theta$.

Let $x', y' \in B'[F^+]$ be such that $[x']\Phi = x$ and $[y']\Phi = y$. Then by Claim 3 $x \equiv y$ (Θ^+) implies $x' \equiv y'$ $(\Theta[F^+] + \Phi)$.

Let $x = f_{\lambda,i}(s)$ and $y = f_{\lambda,i+1}(t)$, let $\lambda = \langle a, b, c, d\rangle$, and recall that $a_0 = d$, $b_0 = c$, and $a_i = a$ and $b_i = b$ for $i = 1, 2, 3$. Since (a) and (b) have been proved, $\Psi(x, y)$ exists. Certainly $x \equiv y$ $((\Psi(x, y))^+)$. Let $x' = f_{\lambda,i}(s)$ in $\mathfrak{B}'[F^+]$ and let $y' = f_{\lambda,i+1}(t)$ in $\mathfrak{B}'[F^+]$. So $x' \equiv y'$ $(\Theta[F^+] + \Phi)$. Let $x' = s_0, \cdots, s_n = y'$ be a shortest sequence with $s_i \equiv s_{i+1}$ $(\Theta[F^+] \cup \Phi)$. Using this sequence and an argument similar to that in Claim 1 one can show that if $i = 2$ or 3, then $x' \equiv f_{\lambda,i}(b_i)$ $(\Theta[F^+])$ and that if $i = 0$ or 1, then $y' \equiv f_{\lambda,i+1}(a_{i+1})$ $(\Theta[F^+])$. Thus in $\mathfrak{B}^+$ we have $y \equiv x \equiv f_{\lambda,i}(b_i) = f_{\lambda,i+1}(a_{i+1})$ (Θ^+), or we have $x \equiv y \equiv f_{\lambda,i+1}(a_{i+1}) = f_{\lambda,i}(b_i)$ (Θ^+). Thus $\Psi(x, y) \subseteq \Theta$.

Let $x \in B$, $y = f_{\lambda,2}(t)$, $\lambda = \langle a, b, c, d\rangle$ and recall that $a_0 = d$, $b_0 = c$, and $a_i = a$ and $b_i = b$ for $i = 1, 2$, or 3. By (b) and (c) $\Psi(a, b)$ exists. Certainly $x \equiv y$ $((\Psi(x, y)^+)$. Let $x' = x$ and let $y' = f_{\lambda,2}(t)$ in $\mathfrak{B}'[F^+]$. So $x' \equiv y'$ $(\Theta[F^+] + \Phi)$. Let $x' = s_0, \cdots, s_n = y'$ be a shortest sequence with $s_i \equiv s_{i+1}$ $(\Theta[F^+] \cup \Phi)$ for $i = 0, \cdots, n-1$. Again using this sequence and an argument similar to that in Claim 1, one can show that $y' \equiv f_{\lambda,2}(a_2)$ $(\Theta[F^+])$. So in $\mathfrak{B}^+$ we have $x \equiv y \equiv f_{\lambda,2}(a_2) = f_{\lambda,1}(b_1)$ (Θ^+). So $\Psi(x, y) \subseteq \Phi$.

Consider (e). Using arguments similar to those in (c) and (d) we get that $x \equiv f_{\lambda,1}(a_1) = f_{\lambda,0}(b_0)$ (Θ^+). The rest follows from (c).

Define $\gamma \colon B^+ \to B$ by

$$\gamma(b^+) = \begin{cases} b^+, & \text{if } b^+ \in B; \\ f_{\lambda,1}(a_1), & \text{if } b^+ \in f_{\lambda,1}(B) \cup f_{\lambda,2}(B); \\ f_{\lambda,3}(b_3), & \text{if } b^+ \in f_{\lambda,3}(B) - f_{\lambda,2}(B). \end{cases}$$

γ is well defined because if $b^+ \in B \cap f_{\lambda,1}(B)$ then $b^+ = f_{\lambda,1}(a_1)$ and if $b^+ \in B \cap f_{\lambda,3}(B)$ then $b^+ = f_{\lambda,3}(b_3)$. By (d) and (c), γ has the required properties.

Consider (g). Let $x' = f_{\lambda,i}(s)$ in $\mathfrak{B}'[F^+]$ and $y' = f_{\mu,j}(t)$ in $\mathfrak{B}'[F^+]$. Starting with $x' = s_0$ and using an argument similar to that in (c) or (d), we conclude there is a $z' \in B$ such that $y' \equiv x' \equiv z'$ $(\Theta[F^+] + \Phi)$. So in $\mathfrak{B}^+$ $y \equiv x \equiv z$ (Θ^+) where $[z']\Phi = z \in B$. (g) now follows from (f).

Claim 7. $\mathcal{H}^+ = \{\Theta^+ \mid \Theta \in \mathcal{H}\}$ is a closure system and $\mathcal{H}^+$ and $\mathfrak{B}^+$ satisfy the hypotheses of Lemma 8.

$\mathcal{K}^+$ is a closure system by Claim 6 and Lemma 7. Condition (8.A) holds for $\mathfrak{B}^+$ with $C = B$. That (8.B) holds follows from Claim 6(f) since $\Theta_{\mathcal{K}^+}(a, b) = (\Phi(a, b))^+$ for all $a, b \in B^+$. Let $f \in F^+$ and $\Theta \in \mathcal{K}$ and $x, y \in B$. Suppose $f(x) \equiv f(y)$ (Θ^+). If $f \in F$, then $f(x) \equiv f(y)$ (Θ). Then by hypothesis $x \equiv y$ (Θ), and so $x \equiv y$ (Θ^+). If $f = f_{\lambda,i}$ for some λ, i, then by Claim 6(b) $x \equiv y$ (Θ), and so $x \equiv y$ (Θ^+).

6. **The theorem.** First we need one more lemma.

Lemma 12. *If $\mathfrak{L}$ is any algebraic lattice, then there is a set C and a unary-algebraic closure system $\mathcal{K}$ of equivalence relations on the set C such that $\mathfrak{L}$ is isomorphic to* $\langle \mathcal{K}; \subseteq \rangle$.

Proof. Let $\mathfrak{C} = \langle C; \vee \rangle$ be the semilattice with zero of compact elements of $\mathfrak{L}$. If I is an ideal of $\mathfrak{C}$, then define the equivalence relation Θ_I on C by $x \equiv y$ (Θ_I) iff $x = y$ or $x, y \in I$. Set $\mathcal{K} = \{\Theta_I \mid I \text{ is an ideal of } \mathfrak{C}\}$, and let $\mathfrak{J}(\mathfrak{C})$ be the lattice of all ideals of $\mathfrak{C}$. It is well known that $\mathfrak{L}$ is isomorphic to $\mathfrak{J}(\mathfrak{C})$, and it is obvious that $\mathfrak{J}(\mathfrak{C})$ is isomorphic to $\langle \mathcal{K}; \subseteq \rangle$.

For each $\langle x, y \rangle \in C^2$ with $x \neq y$ define a partial unary operation $f_{x,y}$ with $D(f_{x,y}) = \{x, y\}$ and $f_{x,y}(x) = x \vee y$ and $f_{x,y}(y) = 0$. For each $x \in C$ define an operation f_x as follows:

$$f_x(y) = \begin{cases} x, & \text{if } x \leq y; \\ 0, & \text{otherwise.} \end{cases}$$

Let F be the set of all these unary partial operations. Set $\mathfrak{C}' = \langle C; F \rangle$. It is left to the reader to show that indeed $\mathcal{C}(\mathfrak{C}')$ does equal $\mathcal{K}$.

Finally we come to the theorem itself.

Theorem. *If $\mathfrak{L}$ is any algebraic lattice, then there is a universal algebra $\mathfrak{A}$ such that $\mathfrak{L}$ is isomorphic to the congruence lattice of $\mathfrak{A}$.*

Proof. Let $\mathfrak{L}$ be an algebraic lattice and let $\mathcal{K}$ and C be given by Lemma 12. Let $\mathcal{K}_0 = \mathcal{K}$ and let $\mathfrak{B}_0 = \langle B_0; F_0 \rangle = \langle C; \{i\} \rangle$ where i is the identity map on C. Suppose $\mathcal{K}_n$ is a unary-algebraic closure system of congruences of the unary partial algebra $\mathfrak{B}_n = \langle B_n; F_n \rangle$, and suppose $f(a) \equiv f(b)$ (Θ) implies $a \equiv b$ (Θ) for any $a, b \in B_n$, $f \in F_n$ and $\Theta \in \mathcal{K}_n$. Let $\mathcal{K}_n^+$ and $\mathfrak{B}_n^+$ be given by Lemma 11. Set $\mathcal{K}_{n+1} = \mathbf{F}(\mathcal{K}_n^+)$ and $\mathfrak{B}_{n+1} = \langle B_{n+1}; F_{n+1} \rangle = \mathbf{F}(\mathfrak{B}_n^+)$. By Lemmas 11, 10, and 6, $\mathcal{K}_{n+1}$ is a unary-algebraic closure system of congruences of $\mathfrak{B}_{n+1}$. By Lemmas 11 and 10 $f(a) \equiv f(b)$ (Θ) implies $a \equiv b$ (Θ) for any $a, b \in B_{n+1}$, $f \in F_{n+1}$ and $\Theta \in \mathcal{K}_{n+1}$.

Notice that $B_0 \subseteq \cdots \subseteq B_n \subseteq \cdots$ and $F_0 \subseteq \cdots \subseteq F_n \subseteq \cdots$. Set $A = \bigcup(B_n \mid n = 0, 1, \cdots)$ and set $F = \bigcup(F_n \mid n = 0, 1, \cdots)$. Since $\mathfrak{B}_{n+1}$ is an expansion of $\mathfrak{B}_n$, we can define an algebra $\mathfrak{A} = \langle A; F\rangle$ in the obvious way, and it will be well defined.

Let $\Psi = \Psi_0 \in \mathcal{H}_0 = \mathcal{H}$. If $\Psi_n \in \mathcal{H}_n$, then set Ψ_{n+1} to be the smallest extension of Ψ_n to a congruence of $\mathfrak{B}_{n+1}$. By Lemmas 11 and 4 Ψ_{n+1} exists, and by definition $\Psi_{n+1} \in \mathcal{H}_{n+1}$. Set $\overline{\Psi} = \bigcup(\Psi_n \mid n = 0, 1, \cdots)$. It is easy to check that $\overline{\Psi}$ is a congruence relation of $\mathfrak{A}$. Now $\overline{\Psi} \cap C^2 = C^2 \cap \bigcup(\Psi_n \mid n = 0, 1, \cdots) = \bigcup(\Psi_n \cap C^2 \mid n = 0, 1, \cdots) = \bigcup(\Psi \mid n = 0, 1, \cdots) = \Psi$. So $\overline{\Psi}$ is an extension of Ψ, and it is clearly the smallest extension of Ψ to a congruence of $\mathfrak{A}$.

Thus $\Psi \to \overline{\Psi}$ is an embedding of $\langle \mathcal{H}; \subseteq\rangle$ into $\mathfrak{C}(\mathfrak{A})$.

Let Θ be a congruence of $\mathfrak{A}$. Obviously $\Theta \cap B_n^2$ is a congruence of $\mathfrak{B}_n$. Note that $\Theta = \bigcup(\Theta \cap B_n^2 \mid n = 0, 1, \cdots)$. By (11.ii) the only congruences on $\mathfrak{B}_n$ that have extensions to congruences of $\mathfrak{B}_{n+1}$ are members of $\mathcal{H}_n$. Since $\Theta \cap B_{n+1}^2$ is an extension of $\Theta \cap B_n^2$, we have $\Theta \cap B_n^2 \in \mathcal{H}_n$. Since $\Theta \cap B_{n+1}^2 \in \mathcal{H}_{n+1}$, it follows that $\Theta \cap B_{n+1}^2$ is the smallest extension of $\Theta \cap B_n^2$ to a congruence of $\mathfrak{B}_{n+1}$. Set $\Psi = \Theta \cap C^2$. Then $\Psi_n = \Theta \cap B_n^2$ and $\overline{\Psi} = \bigcup(\Psi_n \mid n = 0, 1, \cdots) = \bigcup(\Theta \cap B_n^2 \mid n = 0, 1, \cdots) = \Theta$. Thus the embedding $\Psi \to \overline{\Psi}$ is an isomorphism from $\langle \mathcal{H}; \subseteq\rangle$ onto $\mathfrak{C}(\mathfrak{A})$.

7. **Comments.** After going through the preceding "simple" proof, a natural first question is, "Does one really have to go to all that trouble?". The answer given in the first part of this section is a qualified "yes."

If one is going to prove the theorem by constructing an algebra $\mathfrak{A}$ with given congruence lattice $\mathfrak{L}$, then one must find a set A and a system $\mathcal{K}$ of equivalence relations on A. First of all $\mathcal{K}$ must be an algebraic closure system with $\omega \in \mathcal{K}$. Moreover $\langle \mathcal{K}; \subseteq\rangle$ must be a complete sublattice of the partition lattice over A; i.e., join in $\langle \mathcal{K}; \subseteq\rangle$ must be equivalence relation join.

Let $\mathcal{H}$ and C be as given within the proof of Lemma 12. A natural first step would be to try to find an F so that $\mathcal{H}$ was the system of all congruence relations of the (full) algebra $\langle C; F\rangle$. In fact, as is shown below, such an F exists if and only if $\langle \mathcal{H}; \subseteq\rangle$ is a chain. The most obvious problem with $\mathcal{H}$ is that if there are two noncomparable elements, then the join in $\langle \mathcal{H}; \subseteq\rangle$ is not equivalence relation join. There is another related problem which is more subtle. Many constructions appearing in attempted proofs of conjectures involving congruence lattices have this related problem, for example, the one in [10].

Let $\mathfrak{A} = \langle A; F\rangle$ be some (full) algebra. Suppose there exist $a, b \in A$ with $a \neq b$. Suppose for all $x \neq a$ that $a \equiv x$ (Θ) and Θ being congruent implies that $a \equiv b$ (Θ). It is easy to check that $\Theta_{\mathfrak{C}(\mathfrak{A})}(a, b)$ is a nonzero element of $\mathfrak{C}(\mathfrak{A})$ having property (*) where (*) is given below (see e.g. [7]).

(*) If $a \leq \bigvee(x_i \mid i \in I)$, then $a \leq x_i$ for some i.

Now if $x, y \in C$, if $x \neq y$, if $\Theta \in \mathcal{H}$ and if $x \equiv y$ (Θ), then $x \equiv 0$ (Θ) where 0 is the zero of the semilattice $\mathfrak{C} = \langle C; \vee \rangle$. Let $x, y \in C$. Then $\Theta_{\mathcal{H}}(x \vee y, 0) = \Theta_{\mathcal{H}}(x, 0) \vee \Theta_{\mathcal{H}}(y, 0)$. So if there is an F with $\mathcal{H} = \mathcal{C}(\langle C; F \rangle)$, then $(*)$ holds, and so $\Theta_{\mathcal{H}}(x \vee y, 0) = \Theta_{\mathcal{H}}(x, 0)$ or $\Theta_{\mathcal{H}}(y, 0)$. It follows that $x \vee y = x$ or y, and so x and y are comparable in $\mathfrak{C}$. Thus $\mathfrak{C}$, $\mathfrak{L}$ and $\langle \mathcal{H}; \subseteq \rangle$ are all chains. Incidentally, in case $\langle \mathcal{H}; \subseteq \rangle$ is a chain, then letting F consist of the unary (full) operations of the form f_x from the proof of Lemma 12 gives $\mathcal{H} = \mathcal{C}(\langle C; F \rangle)$.

So if one starts with $\mathcal{H}$ and C as given in Lemma 12, one cannot stop there. In general one will need to enlarge the set C.

Theorem 10.3 of [2] shows that the next step in any successful proof involving $\mathcal{H}$ and C is to introduce some sequence of operations having the properties that the sequence $f_{\lambda,1}, f_{\lambda,2}, f_{\lambda,3}$ has in $\mathfrak{B}^+$. Since we need to enlarge the base set C, we might as well allow the sequence of operations that are introduced to have some values lying outside C if it proves to be useful to do so. That one can always introduce such sequences without any troublesome side effects is Lemma 11.

Let $\mathfrak{B}$ be a partial algebra, and let $\mathcal{H}$ be a closure system of congruences of $\mathfrak{B}$. One might at first expect that $\mathbf{F}(\mathcal{H})$ is always a closure system, but it is not. In fact $\mathcal{H}[F]$ or even $\mathcal{H}[f]$ need not be a closure system. For example, let $B = \{0, 1, 2\}$, $f(0) = 2$, $f(1) = 1$ and $D(f) = \{0, 1\}$. Let $\mathfrak{B} = \langle B; \{f\} \rangle$, and let $\mathcal{H}$ be the system of all congruences of $\mathfrak{B}$. Then $f(2) \equiv 2$ $((\Theta_{\mathcal{H}}(0, 2))[f] \cap (\Theta_{\mathcal{H}}(1, 2))[f])$ but $(\Theta_{\mathcal{H}}(0, 2) \cap \Theta_{\mathcal{H}}(1, 2))[f] = \omega$.

Let us assume for the next few paragraphs that the partial operations in $\mathfrak{B}$ have nonempty domains. Since free extension usually does not preserve intersection of congruences, one would wonder if there was not some other extension to an algebra that would preserve intersections. It seems likely that there is in general no such extension because congruences of a smaller extension appear in a dual ideal of $\mathcal{C}(\mathbf{F}(\mathfrak{B}))$. But note that $\langle \mathbf{F}(\mathcal{H}); \subseteq \rangle$ is always a lattice isomorphic to $\langle \mathcal{H}; \subseteq \rangle$. So the only property of $\mathcal{H}$ not preserved in general is that of being a closure system.

Lemma 10 provides sufficient conditions for $\mathbf{F}(\mathcal{H})$ to be a closure system, but it is complicated. A natural conjecture is that if $\mathcal{H}[F]$ is a closure system, then $\mathbf{F}(\mathcal{H})$ is also a closure system. Set $\mathcal{H}[F^0] = \mathcal{H}$ and $\mathcal{H}[F]^{n+1} = (\mathcal{H}[F]^n)[F]$. In fact it appears that for every n there exists a $\mathfrak{B}$ and $\mathcal{H}$ such that $\mathcal{H}[F]^n$ is a closure system and $\mathcal{H}[F]^{n+1}$ is not a closure system. For example, in case $n = 2$ let $\mathfrak{B}$ be the partial unary algebra with $B = \{0, 1, \cdots, 9\}$ and with one unary operation f such that $D(f) = \{0, 1, 2, 3\}$ and $f(i) = i + 5$. Let Θ_0 be the equivalence relation whose nontrivial classes are $\{4, 5\}$, $\{0, 7\}$ and $\{2, 9\}$. Let Θ_1 be the equivalence relation whose nontrivial classes are $\{4, 6\}$, $\{1, 8\}$ and $\{3, 9\}$. Since no class intersects $D(f)$ in more than one point, both Θ_0 and Θ_1 are congruences. Note that $\mathcal{H} = \{\omega, \Theta_0, \Theta_1, \iota\}$ is an algebraic closure system. One can check that $\mathcal{H}[F]$ is also a closure system.

$$4 \equiv 5 = f(0) \equiv f(7) = f(f(2)) \equiv f(f(9)) \ (\Theta_0[F][F]).$$

Also we have

$$4 \equiv 6 = f(1) \equiv f(8) = f(f(3)) \equiv f(f(9)) \ (\Theta_1[F][F]).$$

Thus $\Theta_0[F][F] \cap \Theta_1[F][F] \neq \omega$ but $(\Theta_1 \cap \Theta_0)[F][F] = \omega$. So $\mathcal{H}[F][F]$ is not a closure system.

One might observe that in the above example $(\mathcal{C}(\mathfrak{B}))[F]$ was not a closure system. So one can ask if the conjecture is true in the restricted case when $\mathcal{H} = \mathcal{C}(\mathfrak{B})$. To find the answer to this question appears more difficult.

In most of the lemmas we dealt with a system $\mathcal{H}$ of congruences and with a system of very special extensions of members of $\mathcal{H}$, namely the system of smallest extensions to congruences of the larger partial algebra. If necessary one can prove these lemmas for certain other special systems of extensions.

If $\mathfrak{A}$ is any universal algebra, then $\mathcal{C}(\mathfrak{A})$ is a unary-algebraic closure system. Since this is so, one might as well start his construction with a unary-algebraic closure system as we did here. From the very first comment we know that there are unary-algebraic closure systems that cannot be the system of all congruences of a (full) algebra. If one takes $B = \{0, 1, 2, 3\}$ and lets Θ_0 be the smallest equivalence relation on B collapsing 0 and 1 and Θ_1 be the smallest equivalence relation on B collapsing 2 and 3, then $\mathcal{H} = \{\omega, \Theta_0, \Theta_1, \iota\}$ is an algebraic closure system that is not unary-algebraic. For arbitrary closure systems one could define n-algebraic to mean that a set was closed if and only if it contains the closure of its n-element subsets. In spite of Lemma 5, a unary-algebraic system $\mathcal{H}$ of equivalence relations need not be 1-algebraic. The problem is that a relation containing the $\mathcal{H}$ closure of its singleton's need not be a transitive relation. On the other hand the system $\mathcal{H}$ of the example just above is 2-algebraic but not unary-algebraic. So one might call unary-algebraic $1\frac{1}{2}$-algebraic. Incidentally, unary-algebraic systems accomplish in this proof that which Ω-closed systems accomplish in [9], but the concepts are not in general equivalent. The whole purpose, of course, of introducing either of these concepts is to be able to prove Claim 5 in the proof of Lemma 11.

In the first proofs [2], [3] that appeared for this theorem the emphasis was placed primarily on individual congruences. While in later proofs, [6], [9] or this one, certain systems of congruence relations have been more emphasized.

The set and the system of relations used as the starting point of our construction is different than that used in [2], [3] and [9], and ours is much simpler to deal with. Also, many more operations were used in $\mathfrak{B}_0$ by Grätzer and Schmidt. So the algebras constructed in the proofs differ slightly.

As was pointed out in the introduction to this paper, the newly observed property is stated in condition (C) of Lemma 8. There are two types of expansions of partial algebras used in the constructions in the various proofs. Lemmas 10 and 11 point out that these two expansions both preserve this newly observed property. So the algebra constructed here also has this property. The Grätzer-Schmidt algebra did not have this property because the property does not hold in their starting point of the construction. (Some of their full operations are not even one-to-one.) To surmount this difficulty one need only observe that one can "throw away" all the operations in the starting point of the construction without affecting the congruence lattice of the algebra constructed. An algebra with no operations vacuously satisfies the condition.

Incidentally the computation in Lemma 8 that (C) holds for $\mathfrak{H}[F]$ and $\mathfrak{B}[F]$ could have used (2.ii) in place of (9.ii); i.e., property (C) is always preserved by free extension even if properties (A) and (B) fail.

The main reason for having $D(f, \mathfrak{B}) = C$ for all f in condition (8.A) is to make condition (8.B) easier to state. The lemma is true when one instead simply asserts that $D(f, \mathfrak{B}) \neq \emptyset$ for all f. Of course then in (8.B) one would assert the existence of such a function γ for each set of the form $D(f, \mathfrak{B})$. (One must assert that $D(f, \mathfrak{B}) \neq \emptyset$ so that $\iota[F]$ is again ι. Note that in Lemma 11 $\iota[F^+]$ is not ι but each of $\iota[F^+] + \Phi$ and ι^+ is again ι.)

Based on Lemmas 2 and 3 it is not surprising that Lemmas 8 and 9 are true. But that Lemma 8 is useful seems unnatural at first because (8.C) seems to say that each operation has no effect on the congruences in question. However, as we found out in Lemma 11, if one considers the effect of several such operations together with the effect of transitivity, one can "do" quite a lot with such operations.

For the sake of comparison we list some variations on Lemma 8.

Lemma 8′. *Let $\mathfrak{B} = \langle B; F\rangle$ be a unary partial algebra and let $\mathfrak{H}$ be a closure system at congruences of $\mathfrak{B}$. If*

(A) *there is a $C \subseteq B$, $C \neq \emptyset$, such that $D(f, \mathfrak{B}) = C$ for all $f \in F$,*

(B) *there is a function $\gamma\colon B \to C$ so that $\Theta_{\mathfrak{H}}(b, \gamma(b)) \subseteq \Theta_{\mathfrak{H}}(b, c)$ for all $c \in C$,*

(C′) *for all $a, b \in B - C$ it holds that $\Theta_{\mathfrak{H}}(a, \gamma(a)) \vee \Theta_{\mathfrak{H}}(b, \gamma(b))$ is comparable to $\Theta_{\mathfrak{H}}(a, b)$,*

then $\mathfrak{H}[F]$ is a closure system and $\mathfrak{H}[F]$ and $\mathfrak{B}[F]$ satisfy (A)–(C′).

Lemma 8″. *Let $\mathfrak{B} = \langle B; F\rangle$ be a unary partial algebra and let $\mathfrak{H}$ be a closure system of congruences at $\mathfrak{B}$. If*

(A″) $D(f, \mathfrak{B}) \neq \emptyset$ *for all* $f \in F$,

(B″) *for each* $C = D(f, \mathfrak{B})$ *for some* $f \in F$ *there is a function* $\gamma_C: B \times B \to C$ *so that* $\Theta_{\mathfrak{H}}(b, \gamma(b, a)) \subseteq \Theta_{\mathfrak{H}}(b, c)$ *for any* $a \in B$ *and for any* $c \in C$,

(C″) *there is a* $Q \subseteq B \times B$ *such that*

(a) *if* $\langle a, b\rangle \in Q$; *then* $\langle \gamma_C(a, b), \gamma_C(b, a)\rangle \in Q$ *for any* $C = D(f, \mathfrak{B})$ *for some* f,

(b) $\langle a, b\rangle \in Q$ *and* $a, b \in D(f, \mathfrak{B})$ *for some* f *implies* $\langle f(a), f(b)\rangle \in Q$ *and* $\Theta_{\mathfrak{H}}(a, b) = \Theta_{\mathfrak{H}}(f(a), f(b))$,

(D″) *for each* f *if* $C = D(f, \mathfrak{B})$ *and* $a, b \in B - C$ *then one of the following holds*:

(a) $\Theta_{\mathfrak{H}}(a, b)$ *is comparable to* $\Theta_{\mathfrak{H}}(a, \gamma_C(a, b)) \vee \Theta_{\mathfrak{H}}(b, \gamma_C(b, a))$,

(b) $\langle a, b\rangle \in Q$,

then $\mathfrak{H}[F]$ *is a closure system and* $\mathfrak{H}[F]$ *and* $\mathfrak{B}[F]$ *satisfy* (A″)–(D″).

Lemma 8′ is a refined version of the lemmas used in [6] and [9]. Although somewhat hidden, Lemma 8′ was implicit in the original proof [3]. It was noted in the introduction that the previous techniques for proving this theorem did not apply to nonunary partial algebras. The reason is because the analogue of Lemma 8′ is not true for nonunary partial algebras. (This was not realized initially and some incorrect proofs of some related theorems were given, e.g., [10]. This was pointed out in Example 31 of Chapter 2 of [2].) The analogue of Lemma 8 for arbitrary partial algebras is true.

The analogue of Lemma 8″ for nonunary partial algebras is false. Clearly Lemmas 8 and 8′ are corollaries of Lemma 8″. Also, there exist unary partial algebras satisfying the hypotheses of Lemma 8″ but neither those of 8 nor 8′. Clearly there exist partial algebras satisfying the hypotheses of Lemma 8 but not those of Lemma 8′ and vice versa. Of the three lemmas, Lemma 8 is the easiest to prove and to apply.

Lemma 7 seems a bit unusual because it in essence says that in $B^* \times B^*$ if the closure of any one element set exists, then the closure of every set exists. This lemma was implicit in [3].

Much of Lemma 11 was known before, see [2], [3], [6], [9]. Of course, the two references to condition (8.C) are new to this paper. Also, the concept of unary-algebraic closure systems was applied to Lemma 11 only here and in [6].

Finally, observe that if $\mathfrak{A} = \langle A; F\rangle$ is the algebra constructed here in the proof and $a \in A$, then the subalgebra generated by a is A. This happens because if $b \in A$ and $b \neq a$, then with $\lambda = \langle a, b, b, a\rangle$ there is the operation $f_{\lambda,1}$, and $f_{\lambda,1}(a) = b$. Now set $\mu = \langle a, b, a, b\rangle$. So $f_{\mu,1}(a) = a$. But, by the properties of free extension, $f_{\mu,1}(x) \neq x$ if $x \neq a$. So if σ is an automorphism of $\mathfrak{A}$, then $a\sigma = a$. Since a was arbitrary, σ is the identity map. So the unary algebra $\mathfrak{A}$ constructed here is an algebra with no constant polynomials, one-to-one operations, no subalgebras, and no automorphisms.

Incidentally, one need not assume $\mathfrak{L}$ has two or more elements. All lemmas hold, and $\mathfrak{A} = \mathfrak{B}_0$ is the one element unary algebra with one operation.

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REFERENCES

1. G. Birkhoff and O. Frink, *Representations of lattices by sets*, Trans. Amer. Math. Soc. 64 (1948), 229–316. MR 10, 279.

2. G. Grätzer, *Universal algebra*, Van Nostrand, Princeton, N. J., 1968. MR 40 #1320.

3. G. Grätzer and E. T. Schmidt, *Characterizations of congruence lattices of abstract algebras*, Acta. Sci. Math. 24 (1963), 34–59. MR 27 #1391.

4. G. Grätzer and W. A. Lampe, *On subalgebra lattices of universal algebras*, J. Algebra 7 (1967), 263–270. MR 36 #1377.

5. A. A. Iskander, *The lattice of correspondences of universal algebras*, Izv. Akad. Nauk SSSR Ser. Math. 29 (1965), 1357–1372. (Russian) MR 33 #7289.

6. W. A. Lampe, *On related structures of a universal algebra*, Ph.D. Dissertation, Pennsylvania State University, University Park, Pa., March 1969.

7. ———, *Notes on related structures of a universal algebra*, Pacific J. Math. 43 (1972), 189–205.

8. ———, *Subalgebra lattices of a unary algebra and an axiom of choice*, Colloq. Math. (to appear).

9. E. T. Schmidt, *Kongruenzrelationen algebraischer Strukturen*, Mathematischer Forschungsberichte, 25, Deutscher Verlag der Wissenschaften, Berlin, 1969. MR 41 #136.

10. ———, *Universale Algebren mit gegebenen Automorphismengruppen und Kongruenzverbänden*, Acta. Math. Acad. Sci. Hungar. 15 (1964), 37–45. MR 29 #3411.

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